6.7 Inner Product Spaces.

Consider a vector space V with appropriate definition of addition and scalar multiplication operations.

Define a function

$$\langle , \rangle : V \times V \longrightarrow \mathbb{R}$$
$$(\hat{u}, \vec{v}) \longrightarrow \langle \vec{u}, \vec{v} \rangle$$

Such that

For all $\vec{u}, \vec{v}, \vec{w} \in V$ and all scalars $c \in \mathbb{R}$, the following axioms or properties are satisfied:

1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

2) $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

3) $\langle c\vec{u}, \vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$

4) $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$.

Functions as $\langle , \rangle$ are called Inner Product for a vector space V.

The advantage to have an Inner Product defined in a Vector Space V is that "geometrical" Concepts such as:

a) Length of a Vector

b) distance between Vectors

c) Orthogonality between vectors

d) Approximation of one Vector by another

e) Angle between Vectors

which are so natural or familiar in $\mathbb{R}^2, \mathbb{R}^3$ and $\mathbb{R}^n$ Can be extended to arbitrary Vector spaces V using the Inner Prod.

<u>Definition</u> A vector space V with an Inner Product defined as above is called an <u>"Inner Product Space"</u>.

<u>Definitions</u>

a) <u>Length</u>: $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

b) <u>distance btw $\vec{u}$ and $\vec{v}$</u>: $\|\vec{u}-\vec{v}\| = \sqrt{\langle \vec{u}-\vec{v}, \vec{u}-\vec{v} \rangle}$

c) <u>$\vec{u}$ and $\vec{v}$ are orthogonal</u>: $\langle \vec{u}, \vec{v} \rangle = 0$

d) <u>$\vec{u}$ is a unit vector</u>: $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} = 1 \Leftrightarrow \langle \vec{u}, \vec{u} \rangle = 1$.

Example 1 $\quad t_0, t_1, \ldots, t_n \in \mathbb{R}$

$p, q \in \mathbb{P}_n$ (polynomials of degree $\le n$).

$$\boxed{\langle p,q\rangle \overset{\text{def}}{=} p(t_0)\,q(t_0) + p(t_1)\,q(t_1) + \cdots + p(t_n)\,q(t_n)}$$

Easy to prove that $\langle p,q\rangle$ defined as above is an inner product for $\mathbb{P}_n$. (Satisfy all axioms 1-4).

In particular, consider

$V = \mathbb{P}_2$ and $t_0 = 0, \quad t_1 = \frac{1}{2}, \quad t_2 = 1$

$p(t) = 12t^2, \quad q(t) = 2t - 1$

then

$$\langle p,q\rangle = p(0)\,q(0) + p(1/2)\,q(1/2) + p(1)\,q(1)$$
$$= (0)(-1) + (3)\ (0) + (12)(1) = 12$$

or $\quad \boxed{\langle p,q\rangle = 12}$

Similarly, $\quad \boxed{\langle q,q\rangle = 2}$

Since $\langle q,q\rangle = q(0)q(0) + q(1/2)q(1/2) + q(1)q(1)$
$$= (-1)(-1) + (0)(0) + (1)(1) = 2.$$

lenghts of p and q ?

$$\|p\|^2 = \langle p, p \rangle = (p(0))^2 + (p(\tfrac{1}{2}))^2 + (p(1))^2$$

$$= 0 + 3^2 + 12^2 = 153$$

$\Rightarrow$ lenght of $p = \|p\| = \sqrt{153}$. , $(p(t) = 12t^2)$

Can you find a meaning of this?

$\|q\|^2 = 2 \Rightarrow \|q\| = \sqrt{2}$. $\quad (q(t) = 2t-1)$

Example 2.

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Example 2

Consider the Vector Space

$$C[a,b] = \{ f : [a,b] \to \mathbb{R} : f \text{ is continuous} \}$$

For $f, g \in C[a,b]$, an important inner product is defined as

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_a^b f(x) g(x)\, dx \qquad (5.1)$$

It's not hard to prove that axioms 1-4 are all satisfied by (5.1).

i) Find the length of $f(x) = x^2$ in $x \in [0,2]$.

In general, for $f \in C[a,b]$

$$\|f\| = \left[\int_a^b f^2(x)\, dx \right]^{1/2}$$

So

$$\|f(x) = x^2\| = \left[\int_0^2 (x^2)^2 dx \right]^{1/2} = \left[\frac{x^5}{5} \Big|_0^2 \right]^{1/2}$$

$$= \left(\frac{64}{5} \right)^{1/2} = \frac{8}{\sqrt{5}} = \frac{8\sqrt{5}}{5}.$$

ii) Find the distance between

$$f(x) = x^2 \text{ and } g(x) = x^2+1, \qquad x \in [0,1].$$

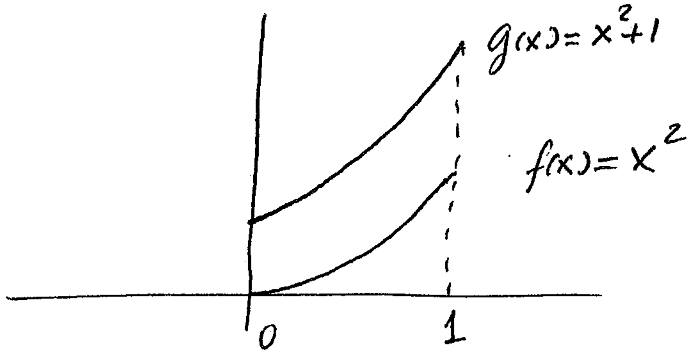
In general,

for $f, g \in C[a,b]$

$$d(f,g) \stackrel{\text{def}}{=} \|f-g\| = \left[\int_a^b (f(x)-g(x))^2 dx\right]^{1/2}$$

In our case,

$$d(f,g) = \left[\int_0^1 [x^2-(x^2+1)]^2 dx\right]^{1/2} = [1]^{1/2} = 1$$



Obviously, this distance between f and g will change if the interval $[a,b]$ changes. This is shown in the next page.

Notice that $d(f,g)$ changes if the interval $[a,b]$ of the space $C[a,b]$ changes.

In fact, if $[a,b] = [-1,1]$

$$d(f,g) = \|f-g\| = \langle f-g, f-g\rangle^{1/2} = \left(\int_{-1}^{1} (x^2+1-x^2)^{1/2} dx\right)^{1/2} =$$

$$= \left(\int_{-1}^{1} dx\right)^{1/2} = \sqrt{2}$$

Some algebraic properties of inner product

Thm $\vec{u}, \vec{v}, \vec{w} \in V$ (inner prod. space) and $k \in \mathbb{R}$, then:

a) $\langle \vec{0}, \vec{v}\rangle = \langle \vec{v}, \vec{0}\rangle = 0$

b), c) $\langle \vec{u}, \vec{v} \pm \vec{w}\rangle = \langle \vec{u}, \vec{v}\rangle \pm \langle \vec{u}, \vec{w}\rangle$

d) $\langle \vec{u}-\vec{v}, \vec{w}\rangle = \langle \vec{u}, \vec{w}\rangle - \langle \vec{v}, \vec{w}\rangle$

e) $k\langle \vec{u}, \vec{v}\rangle = \langle \vec{u}, k\vec{v}\rangle$

Easy to show. Base your proof only on axioms of inner product.

Proof: (a) $\langle \vec{v}, \vec{0}+\vec{0}\rangle = \langle \vec{v}, \vec{0}\rangle$

also $\langle \vec{v}, \vec{0}+\vec{0}\rangle \overset{ax.\,2}{=} \langle \vec{v}, \vec{0}\rangle + \langle \vec{v}, \vec{0}\rangle \Rightarrow \underbrace{\langle \vec{v}, \vec{0}\rangle - \langle \vec{v}, \vec{0}\rangle}_{\vec{0} \text{ real \#s}} = $ $\overset{\text{real \#s}}{=0}$

$= (\langle \vec{v}, \vec{0}\rangle + \langle \vec{v}, \vec{0}\rangle) - \langle \vec{v}, \vec{0}\rangle \overset{\text{rel \#s}}{=} \langle \vec{v}, \vec{0}\rangle + (\langle \vec{v}, \vec{0}\rangle - \langle \vec{v}, \vec{0}\rangle)$

$= \langle \vec{v}, \vec{0}\rangle$ or $\vec{0} = \langle \vec{v}, \vec{0}\rangle$. ✓

Length, Distance and orthogonality properties in General Inner Product Spaces.

Most of the properties already discussed for $\mathbb{R}^n$ with the "dot product" are also Valid for general Inner Products.

For example,

i) Pythagoras Theorem.

ii) Existence of orthogonal bases.

iii) Grand-Smith Process.

iv) Orthogonal Decomposition Theorem.

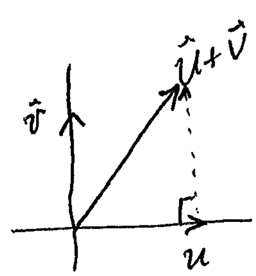
v) Best Approximation Theorem.

For instance,

Theorem (Pythagoras Theorem)

$\vec{u}, \vec{v}$ orthogonal vectors in V (Inner Prod. Space).

Then,
$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$



Proof.-

$$\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v} \rangle =$$
$$= \langle \vec{u},\vec{u} \rangle + 2\underbrace{\langle \vec{u},\vec{v} \rangle}_{=0} + \langle \vec{v},\vec{v} \rangle = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Two Important Inequalities.

Consider a Subspace $W \subset V$ (I.P.S)

For $\vec{v} \in V$, the orthogonal Decomposition Theorem establishes that

$$\vec{v} = \operatorname{proj}_W \vec{v} + (\vec{v} - \operatorname{proj}_W \vec{v})$$

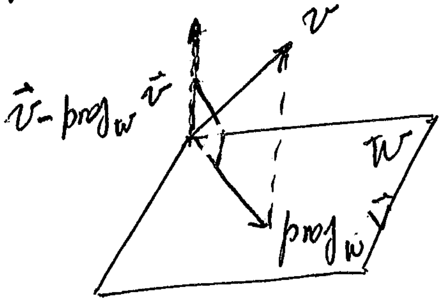
and

$$\langle \vec{v} - \operatorname{proj}_W \vec{v}, \operatorname{proj}_W \vec{v} \rangle = 0$$

Lemma. For any $\vec{v} \in V$ and $\overset{\text{Subspace}}{W} \subset V$ (I.P.S.).

$$\|\operatorname{proj}_W \vec{v}\| \leq \|\vec{v}\|.$$

Proof:



From Pythagoras thm

$$\|\vec{v}\|^2 = \|\operatorname{proj}_W \vec{v}\|^2 + \|\vec{v} - \operatorname{proj}_W \vec{v}\|^2$$

$$\Rightarrow \quad \|\operatorname{proj}_W \vec{v}\| \leq \|\vec{v}\|.$$

Theorem. (Cauchy - Schwarz inequality)

For all $\vec{u}, \vec{v} \in V$ (I.P.S).

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\|.$$

Proof.

If $\vec{u} = \vec{0}$ both sides are zero and the equality is verified.

If $\vec{u} \neq \vec{0}$

$$\|proj_{\vec{u}} \vec{v}\| = \left\| \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \vec{u} \right\| = \left| \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \right| \|\vec{u}\|$$

$$= \frac{|\langle \vec{v}, \vec{u} \rangle|}{\|\vec{u}\|^{\cancel{2}}} \cancel{\|\vec{u}\|} = \frac{|\langle \vec{v}, \vec{u} \rangle|}{\|\vec{u}\|}$$

Since

$$\|proj_{\vec{u}} \vec{v}\| \overset{\text{lemma}}{\leq} \|\vec{v}\|$$

$\Rightarrow$

$$\frac{|\langle \vec{v}, \vec{u} \rangle|}{\|\vec{u}\|} \leq \|\vec{v}\| \Rightarrow \boxed{|\langle \vec{v}, \vec{u} \rangle| \leq \|\vec{u}\| \|\vec{v}\|}$$

Theorem. (Triangle Inequality)

For all $\vec{u}, \vec{v} \in V$ (I.P.S.)

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Proof.

$$\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v} \rangle =$$

$$= \langle \vec{u}, \vec{u} \rangle + 2\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\langle \vec{u}, \vec{v} \rangle \leq$$

$$\leq \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2|\langle \vec{u}, \vec{v} \rangle|$$

Cauchy-Schwarz Ineq.

$$\leq \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\|\vec{v}\|$$

$$= \left(\|\vec{u}\| + \|\vec{v}\|\right)^2$$

$$\Rightarrow \quad \|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|.$$

Consider the Inner Prod. Space $\mathbb{P}_4$ of polynomials of degree ≤ 4, with the Inner Product defined in example 1, but with evaluations at

$$\begin{array}{ccccc} t_0 & t_1 & t_2 & t_3 & t_4 \\ -2 & -1 & 0 & 1 & 2 \end{array}$$

For $p, q \in \mathbb{P}_4$

$$\langle p,q\rangle \overset{def}{=} p(-2)q(-2) + p(-1)\,q(-1) + p(0)\,q(0) + p(1)\,q(1) + p(2)\,q(2)$$

<u>Exercise</u>

a) Check if the set of polynomials $S = \{\overset{p_0}{1}, \overset{p_1}{t}, \overset{p_2}{t^2}\}$ is an orthogonal **set.** If not, apply the Grand-Smith Process to obtain one from it.

b) Find the best approximation to $p(t) = 5 - \frac{1}{2}t^4$ by the **set of** orthogonal polynomials found in part (a).

Answer:

a)

$$p_0(-2)=1,\quad p_0(-1)=1,\quad p_0(0)=1,\quad p_0(1)=1,\quad p_0(2)=1$$

$$p_1(-2)=-2,\quad p_1(-1)=-1,\quad p_1(0)=0,\quad p_1(1)=1,\quad p_1(2)=2$$

$$p_2(-2)=4,\quad p_2(-1)=1,\quad p_2(0)=0,\quad p_2(1)=1,\quad p_2(2)=4$$

Then

$\langle p_0, p_1 \rangle = -2-1+0+1+2=0$ they are orthogonal.

$\langle p_1, p_2 \rangle = -8-1+0+1+8=0$ " " ".

$\langle p_0, p_2 \rangle = 4+1+0+1+4=10$ Not orthogonal.

Since p_0 and p_1 are orthogonal but
p_0 and p_3 are not orthogn

we apply G-S. to obtain a new orthogonal basis

$B_0 = \{ q_0, q_1, q_2 \}$

Where $q_0 = p_0 = 1$
$q_1 = p_1 = t$ and q_2 needs to be computed from G-S process

$$q_2 = p_2 - \frac{\langle p_2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0 - \frac{\langle p_2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1$$

$$= t^2 - \frac{10}{5} 1 - \frac{0}{10} t = t^2 - 2 \Rightarrow \boxed{q_2(t) = t^2 - 2}$$

The new orthogonal set is

$$\boxed{S_0 = \{ \underset{q_0}{1}, \underset{q_1}{t}, \underset{q_2}{t^2-2} \}}$$

$q_2(-2) = 2$ $\quad q_2(1) = -1$
$q_2(-1) = -1$ $\quad q_2(2) = 2$
$q_2(0) = -2$

b) Now, we want to find the best approximation of $p(t) = 5 - \frac{1}{2}t^4$ by the polynomials in S_0.

This is equivalent to find the "projection" of the polynomial $p(t)$ (obviously not a linear combination of q_0, q_1, and q_2) onto the subspace $W = \text{Span}\{1, t, t^2-2\}$.

Then,

$$\boxed{p = \frac{\langle p, q_0 \rangle}{\langle q_0, q_0 \rangle} q_0 + \frac{\langle p, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 + \frac{\langle p, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2}$$

Now,

$p(-2) = -3$, $p(-1) = 9/2$, $p(0) = 5$, $p(1) = 9/2$, $p(2) = -3$

$\langle p, q_0 \rangle = -3 + 9/2 + 5 + 9/2 - 3 = 8$

$\langle p, q_1 \rangle = -6 - 9/2 + 0 + 9/2 + 6 = 0$

$\langle p, q_2 \rangle = -6 - 9/2 - 10 - 9/2 - 6 = -31$

Then,

$$\hat{p} = \frac{8}{5} 1 + 0t + \frac{(-31)}{14}(t^2-2)$$

or $$\boxed{\hat{p}(t) = \frac{8}{5} - \frac{31}{14}(t^2-2)}$$

"Best approximation" of $p(t) = 5 - \frac{1}{2}t^4$ by polynomials in $S_0 = \{1, t, t^2-2\}$.

<u>Exercise</u> For $V = C[0,1]$ with the Inner Product of Example 2, Consider the subspace

$$W = \text{Span}\{\overset{p_1}{1}, \overset{p_2}{2t-1}, \overset{p_3}{12t^2}\}$$

i) Show that the set $S = \{p_1, p_2, p_3\}$ is not an orthogonal set.

ii) Apply G-S process to obtain an orthogonal basis for W.

i) $\langle p_1, p_2 \rangle = \int_0^1 1(2t-1)\,dt = t^2 - t \Big|_0^1 = 0$

$\langle p_2, p_3 \rangle = \int_0^1 (2t-1)\,12t^2\,dt =$

$= \int_0^1 (24t^3 - 12t^2)\,dt = \frac{24t^4}{4} - \frac{12t^3}{3} \Big|_0^1 =$

$= 6t^4 - 4t^3 \Big|_0^1 = 6 - 4 = 2$

So $B = \{1, 2t-1, 12t^2\}$ is not an orthogonal basis for W.

ii) Since p_1 and p_2 are orthogonal. We will keep them in the new orthogonal basis

$q_1 = p_1$ and $q_2 = p_2$

but $\boxed{q_3 = p_3 - \frac{\langle p_3, q_1 \rangle}{\langle q_1, q_1 \rangle = 1} q_1 - \frac{\langle p_3, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2}$

Now, $\langle p_3, q_1 \rangle = \int_0^1 (12t^2)(1)\,dt = 12\frac{t^3}{3} \Big|_0^1 = 4$

$\langle q_2, q_2 \rangle = \int_0^1 (2t-1)^2\,dt = \frac{(2t-1)^3}{6} \Big|_0^1 = \frac{2}{6} = \frac{1}{3}$

Therefore,

$q_3 = 12t^2 - \frac{4}{1}(1) - \frac{2}{\frac{1}{3}}(2t-1) = 12t^2 - 6(2t-1) - 4$
$= 12t^2 - 12t + 2$

New orthog. basis $\boxed{B_0 = \{1, 2t-1, 12t^2 - 12t + 2\}}$